Critical behavior of the spin correlation function in the Ashkin-Teller and Baxter models with a line defect

Carlos Naón
Instituto de Física La Plata, CONICET and Departamento de Fisica, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, CC 67, 1900 La Plata, Argentina
(Received 24 August 2008; revised manuscript received 9 April 2009; published 14 May 2009)

We consider the critical spin-spin correlation function of the Ashkin-Teller and Baxter models. By using path-integral techniques in the continuum description of these models in terms of fermion fields, we show that the correlation decays with distance with the same critical exponent as the Ising model. The procedure is straightforwardly extended to take into account the presence of a line defect. Thus we find that in these altered models the critical index of the magnetic correlation on the defect coincides with the one of the defective two-dimensional Ising or Bariev model.

DOI: 10.1103/PhysRevE.79.051112 PACS number(s): 05.50.+q, 64.60.De, 75.10.Hk

Two-dimensional (2D) statistical-mechanics systems play a central role in our present understanding of phase transitions and critical phenomena. Outstanding members of this family of theories are the Ising model, the Ashkin-Teller (AT) [1], and the eight-vertex (8V) or Baxter models [2]. These last two systems can be mapped onto one another through a duality transformation. They can be considered as two Ising magnets coupled by four-spin interactions. They are the first examples of nonuniversal critical behavior, in the sense that the critical exponents of certain operators are continuous functions of the parameter of the four-spin coupling. An anisotropic version of these models [3,4] leads to an interesting universality-nonuniversality crossover recently analyzed [5]. Apart from academic interest the AT and 8V models are useful to shed light on a variety of phenomena, in both classical and quantum physics, ranging from biological applications [6] to the theory of cuprate superconductors [7].

Concerning the isotropic case, which we will consider in this work, it was conjectured that the magnetization keeps the Ising behavior, with a universal exponent $\Delta_\sigma = 1/8$ [8,9]. This result was later proved by Baxter [4] through corner transfer matrices. It is however surprising that there is no other direct computation of the two-spin correlation function in the literature.

Much less is known exactly about the behavior of these systems in the presence of line defects [10]. For the simpler Ising lattice with an altered row (Bariev model [11]) it has been shown that the scaling index of the magnetization varies continuously with the defect strength [11,12], whereas the critical exponent of the energy density at the defect line remains unchanged [13–15]. Taking this model as working bench, much insight was obtained about the origin of nonuniversal critical behavior. For instance, in Ref. [16] necessary conditions for the dependence of exponents on the coupling constants were derived. Interesting connections with integrable quantum field theories were also revealed [17].

Despite these important advances the behavior of the spin-spin correlator for critical 8V-AT models with line defects remains unknown. The main goal of this paper is to help filling this gap. We shall derive a central feature of that critical behavior through a straightforward calculation performed within the continuous formulation of AT-8V models, using well-established path-integral techniques. Since AT-8V models have both magnetic and electric correlations [4] (the electric correlations have continuously varying exponents [18]), we stress that in this paper we will be concerned with magnetic correlations only. We will show that the magnetic exponent depends on the strength of the defect in exactly the same way as in the Bariev model. In this way, our result provides a very unusual explicit confirmation of universality.

Since it is crucial for our purpose to use a path-integral approach that allows us to factorize the Ising correlator, and this method works for both the usual and altered cases, in order to illustrate it, we start by considering the first case corresponding to the homogeneous lattice. This intermediate step will provide an analytical argument for the behavior of the two-spin correlation function [8,9]. At the end of the paper we will show how the main result, valid for the altered models, is obtained.

The Hamiltonian of the original lattice model is given by

$$\mathcal{H} = -\sum_{\langle ij\rangle} [J_2(\sigma_i\sigma_j + \tau_i\tau_j) + J_4\sigma_i\sigma_j\tau_i\tau_j], \tag{1}$$

where $\langle ij\rangle$ means that the sum runs over nearest neighbors of a square lattice ($\sigma, \tau = \pm 1$).

As shown in Ref. [19] the scaling regime of AT-8V models can be described in the continuum limit in terms of a Thirring-Luttinger Lagrangian, i.e., a model of Dirac fermions coupled by a quartic interaction. Alternatively, this can be expressed as two Majorana fermions interacting via their energy densities as follows:

$$\mathcal{L}[\alpha,\beta] = \bar{\alpha} i \partial\!\!\!/ \alpha + \bar{\beta} i \partial\!\!\!/ \beta - \lambda \epsilon_\alpha \epsilon_\beta, \tag{2}$$

where α and β are the Majorana spinors with components $\alpha_{1,2}$ and $\beta_{1,2}$, respectively. Let us recall that these components are connected to fermion annihilation and creation operators $c_r(d_r)$ and $c_r^\dagger(d_r^\dagger)$ attached to site r $\{c_r = \frac{e^{-i\pi/4}}{\sqrt{2}}[\alpha_1(r) + i\alpha_2(r)]$ and $d_r = \frac{e^{-i\pi/4}}{\sqrt{2}}[\beta_1(r) + i\beta_2(r)]\}$. $\epsilon_\alpha = \alpha_1\alpha_2$ and $\epsilon_\beta = \beta_1\beta_2$ are the corresponding energy densities. The symbol $\partial\!\!\!/$ stands for $\gamma_\mu\partial_\mu$, with γ_μ being the usual Euclidean Dirac matrices ($\mu = 0,1$ associated with space directions). The coupling constant λ is proportional to J_4/J_2.

Similar manipulations, based on the Jordan-Wigner transformation [20], allow us to write the online spin-spin correlation function in the form [21]

$$\langle\sigma(0)\sigma(R)\rangle=\left\langle\exp\left(\pi\int_0^R dx\ \ \epsilon_\alpha(x)\right)\right\rangle, \tag{3}$$

where the vacuum expectation value is an anticommuting path integral to be evaluated with the continuum action $S=\int d^2x\ \mathcal{L}$, with an integration measure $\mathcal{D}\alpha\mathcal{D}\beta$. For $\lambda=0$ the β fields become completely decoupled and the computation can be readily performed either in terms of the Majorana α fields or in terms of Dirac fermions [22] built through the doubling technique [23], yielding the well-known result for the Ising correlator. In what follows we will show how this last procedure can be extended in order to allow for a tractable route leading to the exact computation of the critical exponent of the spin-spin correlation function for 8V and AT models. We start by squaring Eq. (3),

$$\langle\sigma(0)\sigma(R)\rangle^2=\left\langle\exp\left(\pi\int_0^R dx[\epsilon_\alpha(x)+\epsilon_{\alpha'}(x)]\right)\right\rangle, \tag{4}$$

where the vacuum expectation value must now be computed with respect to a Euclidean action with Lagrangian density $\tilde{\mathcal{L}}[\alpha,\beta,\alpha',\beta']=\mathcal{L}[\alpha,\beta]+\mathcal{L}[\alpha',\beta']$, with α' and β' being the replicated fermion fields. Following Ref. [22] we can build Dirac fermions Ψ and χ as the following combinations:

$$\Psi=\alpha+i\alpha',\quad \chi=\beta+i\beta'. \tag{5}$$

In terms of these new fields we can write the Lagrangian $\tilde{\mathcal{L}}[\alpha,\beta,\alpha',\beta']$ in the form

$$\tilde{\mathcal{L}}[\Psi,\chi]=\bar{\Psi}i\partial\!\!\!/\Psi+\bar{\chi}i\partial\!\!\!/\chi-\frac{\lambda}{8}[\bar{\chi}\gamma_5\chi\bar{\Psi}\gamma_5\Psi+\mathrm{Im}(\chi^T\gamma_1\chi^T)\mathrm{Im}(\Psi^T\gamma_1\Psi^T)], \tag{6}$$

where $\gamma_5=i\gamma_0\gamma_1$ and Ψ^T,χ^T are the transposed spinors. On the other hand Eq. (4) can be expressed as

$$\langle\sigma(0)\sigma(R)\rangle^2=\left\langle\exp\left(\pi\int d^2x\ \bar{\Psi}A\!\!\!/\Psi\right)\right\rangle, \tag{7}$$

where now the integration measure in the right-hand side is expressed in terms of the fields Ψ and χ, and A_μ is an auxiliary vector field with components

$$A_0(x_0,x_1)=\delta(x_0)\theta(x_1)\theta(R-x_1),\quad A_1(x_0,x_1)=0. \tag{8}$$

At this point we note that a similar manipulation has been earlier introduced [24] and employed to compute several correlation functions in both the Ising and 8V models [25]. In those cases this method allowed us to identify the objects to be computed with certain fermionic determinants that could be evaluated through an appropriate change of path-integral variables. At first sight one sees that the problem is much more involved in the present case, since the correlation is not directly associated with a simple fermionic determinant. Indeed, gathering the above results we can write

$$\langle\sigma(0)\sigma(R)\rangle^2=\frac{Z[g=\pi]}{Z[g=0]}, \tag{9}$$

where

$$Z[g]=\int\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}\bar{\chi}\mathcal{D}\chi\exp\left(-\int d^2x[\tilde{\mathcal{L}}(\Psi,\chi)-g\bar{\Psi}A\!\!\!/\Psi]\right). \tag{10}$$

This is our first nontrivial result. The continuum limit of the squared two-point spin correlation function is *exactly* expressed in terms of the vacuum to vacuum functional of a quantum field theory describing two interacting fermion species. We now show how the right-hand side of Eq. (9) can be put as the product of two R-dependent factors, one of which being the squared spin-spin correlator of the Ising model. To this end we make the following change in path-integral variables in the numerator of Eq. (9), with chiral and gauge parameters Φ and η, respectively:

$$\Psi=e^{-\pi(\gamma_5\Phi+i\eta)}\zeta,\quad \bar{\Psi}=\bar{\zeta}e^{-\pi(\gamma_5\Phi-i\eta)}. \tag{11}$$

Note that the χ fields are left unchanged. If the parameters of the transformation are related to the previously introduced vector field A_μ in the form

$$A_\mu=\epsilon_{\mu\nu}\partial_\nu\Phi+\partial_\mu\eta, \tag{12}$$

then the only R-dependent term in the action (i.e., the one with "coupling constant" $g=\pi$) becomes completely decoupled. As the result of the change the dependence on R reappears in two places: in the λ term that couples both fermion species χ and ζ, through the dependence of Φ and η on R, and in the Jacobian associated with Eq. (11). As explained in Ref. [26], this Jacobian must be computed with a gauge-invariant regularization prescription in order to avoid an unphysical linear divergence. Following this procedure it has been shown that the Jacobian exactly coincides with the squared critical spin-spin function of the Ising model [24]. Then we have arrived at the following identity:

$$\langle\sigma(0)\sigma(R)\rangle^2=\langle\sigma(0)\sigma(R)\rangle^2_{\mathrm{Ising}}F(\lambda,R), \tag{13}$$

with

$$F(\lambda,R)=\mathcal{N}(\lambda)\langle\exp[S_\Phi(\zeta,\chi)+S_\eta(\zeta,\chi)]\rangle_0, \tag{14}$$

where $\langle\ \rangle_0$ means vacuum expectation value with respect to the model of free χ and ζ fermions. $\mathcal{N}(\lambda)$ is a normalization constant independent of R. Since the analysis of the dependence of $F(\lambda,R)$ on R is more easily done in momentum space, we have Fourier transformed $S_\Phi(\zeta,\chi)$ and $S_\eta(\zeta,\chi)$ in the above equation,

$$S_\Phi(\zeta,\chi)=\frac{\lambda}{8}\int\prod_{j=1}^4\frac{d^2p_j}{(2\pi)^2}\bar{\chi}(p_1)\gamma_5\chi(p_2)\bar{\zeta}(p_3)\gamma_5G(P)\zeta(p_4), \tag{15}$$

with $G(P,R)$ being a diagonal 2×2 matrix given by

$$G(P,R) = \begin{pmatrix} g_+(P,R) & 0 \\ 0 & g_-(P,R) \end{pmatrix}, \tag{16}$$

where $g_\pm(P,R) = \pm \int d^2x\, e^{iPx} e^{\mp 2\pi\Phi(x,R)}$ and $P = p_1 + p_2 + p_3 + p_4$. A similar expression is obtained for S_η with $G(P,R)$ replaced with

$$H(P,R) = \begin{pmatrix} h(P,R) & 0 \\ 0 & h(P,R) \end{pmatrix}, \tag{17}$$

with $h(P,R) = \int d^2x\, e^{iPx} e^{2i\pi\eta(x,R)}$.

The explicit functional forms of $\Phi(x,R)$ and $\eta(x,R)$ can be determined by combining Eqs. (8) and (12), which yields $\Phi(x,R) = -\frac{1}{4\pi} \ln \frac{x_0^2 + (R-x_1)^2 + a^2}{x_0^2 + x_1^2 + a^2}$ and $\eta(x,R) = \frac{x_0}{2\pi} \int_0^R \frac{dy}{x_0^2 + (y-x_1)^2 + a^2}$, where a is an ultraviolet cutoff which can be identified with the lattice spacing of the original discrete system.

Our problem is now reduced to the analysis of the integrals $g_\pm(P,R)$ and $h(P,R)$. In so doing one notes that the integrals diverge for large distances, which leads us to the introduction of a cutoff L which can be interpreted as the size of the system (of course, the thermodynamic limit will be recovered by setting $L \to \infty$ at the end of the computation). In terms of the dimensionless variables $u_\rho = \frac{x_\rho}{L}$ ($\rho = 0,1$), we obtain

$$\begin{aligned} g_\pm(P,R) &= \lim_{L\to\infty} \pm L^2 \int_{|u_\mu|<1} d^2u \\ &\quad \times e^{iLPu} \left(\frac{u_0^2 + [u_1 - (R/L)]^2 + a^2/L^2}{u_0^2 + u_1^2 + a^2/L^2} \right)^{1/2} \\ &= \pm (2\pi)^2 \delta^2(P) \end{aligned} \tag{18}$$

and a similar result for $h(P,R)$. Then, it is apparent now that in the thermodynamic limit ($a \ll R \ll L$) $F(\lambda,R)$ becomes independent of R and the critical behavior exactly coincides with the one of the 2D Ising model. This provides an analytical argument for the conjectures first given in Refs. [8,9].

Let us now address the main issue of this work. We include a line defect in one of the original Ising lattices, say the one with spins σ. To be specific we consider the so-called chain defect (here we employ the terminology of Ref. [10], which corresponds to Bariev's second-type defect, in which bonds along the same column are replaced: $J_2 \to J_2'$). We will study the two-spin correlation function in the column of altered bonds ($x_0 = 0$) [12]. In the following we recall that this model could be mapped on to an *XYZ* spin 1/2 quantum chain [4]. In this framework, seeing x_0 as the temporal variable associated with the quantum evolution, our defect maps onto an additional interaction affecting the whole chain (in contrast to a line defect parallel to the time axis that would map onto an individual impurity site). It is known that the continuous version of the classical model is modified, due to the defect, by the addition in Eq. (2) of a term $2\pi\mu\delta(x_0)\epsilon_\alpha(x)$, with $\mu = J_2' - J_2$. By carefully examining the fermionic representation of σ-spin operators on the lattice, following the lines of Ref. [22], one also finds that in the continuum limit each spin operator on the defect line picks up a similar μ-dependent factor, in such a way that the squared correlator for the defective model is given by a simple modification of Eqs. (7) and (8),

$$\langle \sigma(0)\sigma(R) \rangle_\mu^2 = \left\langle \exp\left[\pi(1+4\mu) \int d^2x\, \bar{\Psi} A\!\!\!/ \Psi\right] \right\rangle_\mu. \tag{19}$$

At this point we stress that this formula is valid for σ spins. The μ dependence of the coefficient comes from the fact that the additional defect term mentioned above can be interpreted as a position-dependent mass term for Ψ fermions. This additional mass term is not present for χ fermions and, consequently, the corresponding coefficient in the string representation for $\langle \tau(0)\tau(R) \rangle_\mu^2$ will be just equal to π. From now on one can follow exactly the same steps already explained for the defect-free case. The presence of the defect manifests in the exponent of the path-integral change in variables given by Eq. (11), where one has to make the substitution $\pi \to \pi(1+4\mu)$. Once again the squared two-point function factorizes in the form

$$\langle \sigma(0)\sigma(R) \rangle_\mu^2 = \langle \sigma(0)\sigma(R) \rangle_{\text{Bariev}}^2 F(\lambda,\mu,R). \tag{20}$$

For the first factor in the right-hand side of this equation we obtain, through the corresponding Jacobian, the well-known behavior for the Bariev model: $\langle \sigma(0)\sigma(R) \rangle_{\text{Bariev}} \simeq (\frac{a}{R})^{2\Delta_\sigma}$, with $\Delta_\sigma = \frac{1}{8}(1+4\mu)^2$ [11,12,27]. Concerning the second factor, $F(\lambda,\mu,R)$ has the same structure as $F(\lambda,R)$, already depicted in Eqs. (14), (16), and (17) [in fact it satisfies $F(\lambda,0,R) = F(\lambda,R)$]. The only difference is the appearance of the μ-dependent factors in the exponents of the integrals defining the functions $g_\pm(P,R)$ and $h(P,R)$. Therefore, it turns out that these integrals can be written following the same prescription as in the $\mu = 0$ case. The corresponding expression coincides with Eq. (18), with an exponent $\kappa = \frac{1}{2}(1+4\mu)$ instead of $\frac{1}{2}$ in the fraction of functions. A similar result is obtained for $h(P,R)$. Let us mention that the constant κ satisfies $|\kappa| < 1$. Indeed, as shown in Ref. [15], the allowed range of defect strengths in the lattice Ising model corresponds to $|\mu| < \frac{1}{4}$ in its continuous version. Since for $\lambda = 0$ we should reobtain the results corresponding to the Bariev model, we conclude that for the present models the above condition also coincides with the physically relevant interval of defect strengths. But the main point is that in the thermodynamic limit ($a \ll R \ll L$), we find that $F(\lambda,\mu;R)$ becomes independent of R, and the critical exponent for the magnetic correlation is unchanged by the coupling between the different Ising subsystems, keeping the same value as in the 2D Ising model with a defect line. This is our main result. It is worth noting that the irrelevance of the λ coupling is consistent with a scaling picture, similar to the one presented in [16]. Indeed, if one sees the coupling on the defect as a perturbation, its scaling dimension is $d - 1 - 2\Delta_\epsilon$, where $\Delta_\epsilon = \frac{1}{1+2\lambda/\pi}$ is the dimension of the energy density. Since $d = 2$ one finds that the coupling is irrelevant for $|\lambda| < \pi/2$.

Concerning the correlations between τ spins, the computation is more subtle. Since τ spins are related to χ fermions, one has to perform transformation (11) not only for the ζ fields, but for the χ fields also. However, due to the fact that the defect only affects the couplings between σ spins, the string representation for τ correlations on the defect line, $\langle\tau(0)\tau(R)\rangle_{\mu}^{2}$ does not pick up any μ-dependent coefficient, in contrast to what happened for $\langle\sigma(0)\sigma(R)\rangle_{\mu}^{2}$ [see Eq. (19)]. One then finds that these correlations decay as in the homogeneous Ising model.

To summarize, we have determined the critical behavior of the two-spin correlation in the continuum field-theory version of isotropic AT and 8V models. This scheme treats in a unified way both the homogeneous (defect free) and inhomogeneous (with a line defect) cases. In the first case we provided an analytical derivation for the value of the magnetic exponent, $\Delta_{\sigma}=\frac{1}{8}$. In the second case we found that on the line defect the critical index maintains Bariev's value $\Delta_{\sigma}=\frac{1}{8}(1+4\mu)^{2}$, where μ is the strength of the defect. The critical index corresponding to the spins with homogeneous couplings remains equal to the Ising value in all cases ($\Delta_{\tau}=\frac{1}{8}$). Our result is an explicit confirmation of universality, in the sense that the interactions between the elementary Ising subsystems do not affect the critical exponents, even in the presence of a line defect. Besides these results, we hope that our approach will be useful to shed light on interesting problems concerning inhomogeneous AT-8V models. In particular it could be used to analyze scaling properties at interfaces between critical subsystems [28].

The author is grateful to CONICET and UNLP (Argentina) for financial support. He thanks D. Cabra, G. Rossini, and M. Salvay for useful discussions and A. Iucci for helpful correspondence.

[1] J. Ashkin and E. Teller, Phys. Rev. **64**, 178 (1943).
[2] R. Baxter, Phys. Rev. Lett. **26**, 832 (1971).
[3] L. P. Kadanoff, Phys. Rev. Lett. **39**, 903 (1977).
[4] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1982).
[5] A. Giuliani and V. Mastropietro, Phys. Rev. Lett. **93**, 190603 (2004).
[6] Z. Chang, P. Wang, and Y.-H. Zheng, Commun. Theor. Phys. **49**, 525 (2008).
[7] Vivek Aji and C. M. Varma, Phys. Rev. Lett. **99**, 067003 (2007).
[8] M. N. Barber and R. J. Baxter, J. Phys. C **6**, 2913 (1973).
[9] L. P. Kadanoff and A. C. Brown, Ann. Phys. (N.Y.) **121**, 318 (1979); H. J. F. Knops, *ibid.* **128**, 448 (1980); M. Kohmoto, M. den Nijs, and L. P. Kadanoff, Phys. Rev. B **24**, 5229 (1981).
[10] F. Iglói, I. Peschel, and L. Turban, Adv. Phys. **42**, 683 (1993).
[11] R. Z. Bariev, Sov. Phys. JETP **50**, 613 (1979).
[12] B. M. McCoy and J. H. H. Perk, Phys. Rev. Lett. **44**, 840 (1980).
[13] A. C. Brown, Phys. Rev. B **25**, 331 (1982).
[14] L. F. Ko, H. Au-Yang, and J. H. H. Perk, Phys. Rev. Lett. **54**, 1091 (1985).
[15] T. W. Burkhardt and Je-Young Choi, Nucl. Phys. B **376**, 447 (1992).
[16] T. W. Burkhardt and E. Eisenriegler, Phys. Rev. B **24**, 1236 (1981).
[17] G. Delfino, G. Mussardo, and P. Simonetti, Nucl. Phys. B **432**, 518 (1994); Phys. Lett. B **328**, 123 (1994).
[18] L. F. Ko and B. M. McCoy, Phys. Rev. Lett. **56**, 2645 (1986); Phys. Lett. A **118**, 287 (1986); L.-F. Ko, Nucl. Phys. B **295**, 571 (1988).
[19] A. Luther and I. Peschel, Phys. Rev. B **12**, 3908 (1978).
[20] T. D. Schultz, D. C. Mattis, and E. H. Lieb, Rev. Mod. Phys. **36**, 856 (1964).
[21] Myron Bander and C. Itzykson, Phys. Rev. D **15**, 463 (1977).
[22] J. B. Zuber and C. Itzykson, Phys. Rev. D **15**, 2875 (1977).
[23] R. A. Ferrell, J. Stat. Phys. **8**, 265 (1973).
[24] C. M. Naón, J. Phys. A **22**, L207 (1989).
[25] V. I. Fernández and C. M. Naón, J. Phys. A **30**, 1843 (1997).
[26] C. M. Naón, J. Phys. A **23**, L273 (1990).
[27] D. C. Cabra and C. M. Naón, Mod. Phys. Lett. A **9**, 2107 (1994).
[28] P. Lajkó, L. Turban, and F. Iglói, Phys. Rev. B **76**, 224423 (2007).